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Graceful Chromatic Numbers of Three Variants of Triangular Ladder Graphs

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ABSTRACT
A graceful k-coloring is a proper vertex coloring with colors in $\{1, 2, \dots, k\}$ whose absolute differences induce a proper edge coloring. We determine the exact graceful chromatic numbers of three triangular ladder variants. For TLX_n, the values are 5 for $n = 2$, 7 for $3 \leq n \leq 7$, and 8 for $n \geq 8$. For $TLXJ_n$, they are 6 for $n = 2$, 7 for $3 \leq n \leq 5$, and 8 for $n \geq 6$. For CTL_n with $n \geq 3$, the value is 7 when $n \equiv 0 \ (mod\ 3)$ and 8 otherwise.

KEYWORDS
Graceful Chromatic Number, Graceful Coloring, Triangular Ladder Graphs, Bridge Triangular Ladder Graphs, Circular Triangular Ladder Graph

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A. Introduction

Throughout this paper, $\boldsymbol{G} = (\boldsymbol{V}(\boldsymbol{G}), \boldsymbol{E}(\boldsymbol{G}))$ denotes a finite, simple, connected graph, where $\boldsymbol{V}(\boldsymbol{G})$ and $\boldsymbol{E}(\boldsymbol{G})$ are its vertex set and edge set, respectively. Graph labeling assigns labels, usually integers, to the vertices or edges of a graph, or to both, subject to specified conditions. Numerous types of graph labelings have been introduced and studied in graph theory (Gallian, 2018). Graceful labeling was introduced by Rosa under the name $\boldsymbol{\beta}$-valuation (Rosa, 1967). The term graceful labeling was subsequently introduced by Golomb (Golomb, 1972). Let $\boldsymbol{q} = |\boldsymbol{E}(\boldsymbol{G})|$. A graceful labeling of $\boldsymbol{G}$ is an injective function $\boldsymbol{f}: \boldsymbol{V}(\boldsymbol{G}) \rightarrow \{\boldsymbol{0}, \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{q}\}$ such that the induced edge-labeling function $\boldsymbol{f}^*: \boldsymbol{E}(\boldsymbol{G}) \rightarrow \{\boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{q}\}$, defined by $\boldsymbol{f}^*(\boldsymbol{uv}) = |\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})|$ for every $\boldsymbol{uv} \in \boldsymbol{E}(\boldsymbol{G})$, is bijective (Gallian, 2018). A graph that admits a graceful labeling is called a graceful graph.

Another fundamental concept in graph theory, in addition to graph labeling, is graph coloring. Motivated by graceful labeling, graceful coloring uses the same absolute difference rule to induce edge colors from the vertex colors, while requiring both the vertex coloring and induced edge coloring to be proper. A proper vertex coloring assigns colors to the vertices of a graph so that adjacent vertices receive different colors. Similarly, a proper edge coloring assigns colors to edges so that any two edges sharing a common endpoint receive different colors (Chartrand & Zhang, 2009). For an integer $k \geq 2$, a graceful k-coloring of G is a proper vertex coloring $f: V(G) \rightarrow \{1, 2, \dots, k\}$ that induces a

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proper edge coloring $f^*: E(G) \to \{1,2,\dots,k-1\}$, defined by $f^*(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ (Bi et al., 2017). The graceful chromatic number of G, denoted by $\chi_g(G)$, is the minimum integer k for which G admits a graceful k-coloring (Laavanya & Yamini, 2024). For a vertex $u \in V(G)$, its degree in G, denoted by $d_G(u)$, is the number of edges incident with u. The maximum degree of G, denoted by $\Delta(G)$, is defined by

$$\Delta(G) = max\{d_G(u) : u \in V(G)\}$$

(Bondy & Murty, 1976).

The graceful chromatic number is closely related to several classical graph-coloring parameters. Let $\chi(G)$ denote the chromatic number of G, $\chi'(G)$ its chromatic index, and $\Delta(G)$ its maximum degree. Bi et al. (2017) established a general lower bound for the graceful chromatic number of a connected graph.

Theorem 1.1 (Bi et al., 2017). Let G be a nontrivial connected graph of order at least three. Then

$$\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1.$$

Since $\chi'(G) \geq \Delta(G)$, it follows that

$$\chi_g(G) \geq \Delta(G) + 1.$$

The graceful chromatic numbers of several well-known graphs and graph classes have been investigated. Mincu et al. (2019) determined the graceful chromatic numbers of the diamond graph, Petersen graph, Moser spindle graph, Goldner-Harary graph, friendship graphs, fan graphs, and several other graph classes. Alfarisi et al. (2019) investigated the graceful chromatic number of several unicyclic graphs. Further results for book graphs and star-related graphs were obtained by Asy'ari et al. (2022). More recently, Kristiana et al. (2023) determined the graceful chromatic numbers of several grid graph families, while Lestari et al. (2024) investigated the graceful chromatic numbers of centripetal graph families. These studies demonstrate that the graceful chromatic numbers depend strongly on the structural properties of the graph being considered. Suparta et al. (2023) also established exact values and bounds for the graceful chromatic numbers of Cartesian products of paths and cycles.

This study investigates the exact graceful chromatic numbers of three variants of triangular ladder graphs. The ladder graph L_n is defined as the Cartesian product $P_n \square P_2$, where P_n is a path of order n (Chartrand & Zhang, 2009). Several modifications of ladder graphs have been introduced, including triangular, diagonal, circular, variation triangular, and bridge triangular ladder graphs. This study focuses on the variation triangular ladder graph and its related modifications.

The variation triangular ladder graph, denoted by TLX_n, was introduced by Atmadja and Sugeng (2017). Its vertex set is defined as $V(TLX_n) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n-1\}$, and its edge set is $E(TLX_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq n-1\} \cup \{w_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iw_i : 1 \leq i \leq n-1\} \cup \{w_iu_{i+1} : 1 \leq i \leq n-1\}$. Consequently, TLX_n has $|V(TLX_n)| = 3n-1$ vertices and $|E(TLX_n)| = 6n-5$ edges. In this definition, no edge joins v_i to v_j for distinct i and j; in particular, the vertices $v_1, \dots, v_n$ form an independent set. The structure of TLX_n is illustrated in Figure 1.

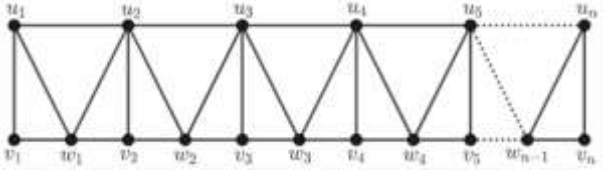


Figure 1. The variation triangular ladder graph TLX_n

The bridge triangular ladder graph, denoted $TLXJ_n$, for $n \geq 2$, is obtained by adding a vertex and its incident edges at each end of the variation triangular ladder graph (Atmadja & Marhaeni, 2020). The vertex set of $TLXJ_n$ is defined as $V(TLXJ_n) = \{u_i, w_i : 1 \leq i \leq n\} \cup \{v_j : 1 \leq i \leq n+1\}$, and its edge set is $E(TLXJ_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i w_i : 1 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{v_i w_i : 1 \leq i \leq n\} \cup \{u_i v_{i+1} : 1 \leq i \leq n\} \cup \{w_i v_{i+1} : 1 \leq i \leq n\}$. Thus, $TLXJ_n$ has $|V(TLXJ_n)| = 3n + 1$ vertices and $|E(TLXJ_n)| = 6n - 1$ edges. The structure of $TLXJ_n$ is illustrated in Figure 2.

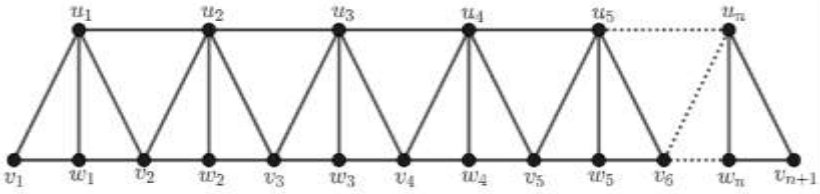


Figure 2. The bridge triangular ladder graph $TLXJ_n$

For $n \geq 3$, the circular triangular ladder graph, denoted by CTL_n is obtained from TL_n by adding the edges $v_1 v_n, u_1 u_n$ and $u_n v_1$ (Lestari, 2022). Its vertex set is $V(CTL_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and its edge set is $E(CTL_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 u_n, v_1 v_n, u_n v_1\}$. Consequently, $|V(CTL_n)| = 2n$ vertices and $|E(CTL_n)| = 4n$ edges. The structure of CTL_n is illustrated in Figure 3.

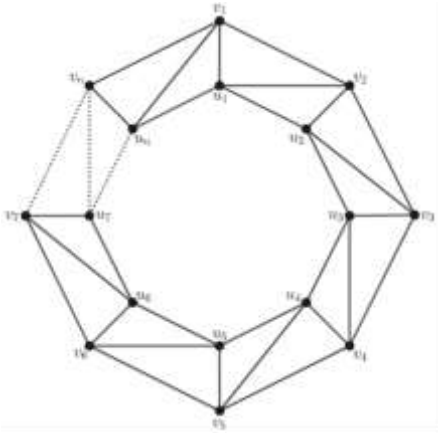


Figure 3. The circular triangular ladder graph CTL_n

The graceful chromatic numbers of several ladder graph classes were determined by Laavanya and Devi Yamini (2024), as summarized in the following theorems.

Theorem 1.2 (Laavanya & Devi Yamini, 2024). For the ladder graph L_n,

$$\chi_g(L_n) = \begin{cases} 4, & if\ n = 2 \\ 5, & if\ n \geq 3. \end{cases}$$

Theorem 1.3 (Laavanya & Devi Yamini, 2024). For the slanting ladder graph SL_n, where $n \geq 4$, $\chi_g(SL_n) = 5$.

Theorem 1.4 (Laavanya & Devi Yamini, 2024). For the triangular ladder graph TL_n,

$$\chi_g(TL_n) = \begin{cases} 6, & if\ n = 3,4 \\ 7, & if\ n \geq 5. \end{cases}$$

Theorem 1.5 (Laavanya & Devi Yamini, 2024). For the diagonal ladder graph DL_n,

$$\chi_g(DL_n) = \begin{cases} 8, & if\ n = 5{,}6 \\ 9, & if\ n \geq 7. \end{cases}$$

Theorem 1.6 (Laavanya & Devi Yamini, 2024). For the circular ladder graph CL_n, where $n \geq 4$,

$$\chi_g(CL_n) = \begin{cases} 5, & if\ n \equiv 0(mod\ 4) \\ 6, & otherwise \end{cases}$$

B. Method

This study employs an axiomatic-deductive approach. For each graph family, the vertex set, edge set, adjacency relations, vertex degrees, and maximum degrees are first identified. Lower bounds for the graceful chromatic numbers are established using structural properties and the bound $\chi_g(G) \geq \Delta(G) + 1$. Matching upper bounds are obtained by constructing explicit vertex colorings and verifying that adjacent vertices receive distinct colors and that all edges incident with a common vertex receive pairwise distinct induced colors.

C. Results and Discussion

Theorem 3.1. For every integer $\boldsymbol{n \geq 2}$, the graceful chromatic number of the variation triangular ladder graph $\boldsymbol{TLX_n}$ is

$$\boldsymbol{\chi_g(TLX_n)} = \begin{cases} \boldsymbol{5}, & \boldsymbol{n = 2} \\ \boldsymbol{7}, & \boldsymbol{3 \leq n \leq 7} \\ \boldsymbol{8}, & \boldsymbol{n \geq 8} \end{cases}$$

Proof. We prove the result by establishing matching lower and upper bounds in each of the following three cases.

Case 1: $n = 2$.

The graph TLX_2 has five vertices and seven edges, with $V(TLX_2) = \{u_1, u_2, v_1, v_2, w_1\}$ and $E(TLX_2) = \{u_1u_2, u_1v_1, u_2v_2, u_1w_1, v_1w_1, w_1u_2, w_1v_2\}$. Since $\deg(w_1) = 4$, we have $\Delta(TLX_2) = 4$. Therefore, by the lower-bound theorem

$$\chi_g(TLX_2) \geq \Delta(TLX_2) + 1 = 5.$$

For the upper bound, define $f: V(TLX_2) \to \{1, 2, 3, 4, 5\}$ by

$$f(u_1) = 4, \quad f(u_2) = 2, \quad f(v_1) = 3, \quad f(v_2) = 5, \quad f(w_1) = 1$$

The induced edge colors are

uv	u_1u_2	u_1v_1	u_2v_2	u_1w_1	v_1w_1	w_1u_2	w_1v_2
$f^*(u,v)$	2	1	3	3	2	1	4

At every vertex, the induced colors of its incident edges are pairwise distinct. Moreover, adjacent vertices receive different colors. Thus, f is a graceful 5-coloring of TLX_2, which implies $\chi_g(TLX_2) \leq 5$. Combining the lower and upper bounds, we conclude that $\chi_g(TLX_2) = 5$.

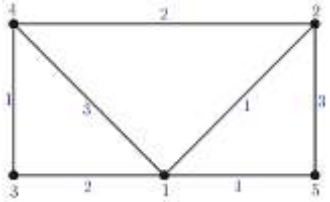


Figure 4. A graceful 5-coloring of TLX_2

Case 2: $3 \leq n \leq 7$

Since $TLX_3 \subseteq TLX_n$ for every $n \geq 3$, it suffices to prove that $\chi_g(TLX_3) \geq 7$. Suppose that TLX_3 admits a graceful 6-coloring f. Since $\deg(u_2) = 5$, the five edges incident with u_2 must receive the five distinct induced colors $1, 2, 3, 4, 5$; hence, $f(u_2) \in \{1, 6\}$. Replacing every color c by $7 - c$, if necessary, preserves all induced edge colors, so we may assume that $f(u_2) = 1$. Therefore, its neighbors u_1, w_1, v_2, w_2, u_3 must receive the colors $2, 3, 4, 5, 6$ in some order. None of these arrangements can be extended simultaneously to both v_1 and v_3. Hence, TLX_3 does not admit a graceful 6-coloring. Therefore, $\chi_g(TLX_3) \geq 7$.

For the upper bound, define $f: V(TLX_n) \to \{1, 2, \ldots, 7\}$, for $3 \leq n \leq 7$, by

i	1	2	3	4	5	6	7
$f(u_i)$	4	2	6	7	2	6	4
$f(v_i)$	1	7	4	1	4	3	5

and

i	1	2	3	4	5	6
$f(w_i)$	5	1	3	5	1	7.

For each n, only the entries satisfying $1 \leq i \leq n$ for u_i, v_i and $1 \leq i \leq n - 1$ for w_i are used. Direct verification shows that adjacent vertices receive distinct colors and that the induced colors of edges incident with each vertex are pairwise distinct. Hence, f is a graceful 7-coloring, so $\chi_g(TLX_n) \leq 7$. Therefore, $\chi_g(TLX_n) = 7$ for $3 \leq n \leq 7$.

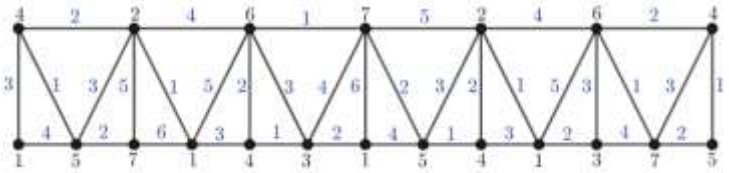


Figure 5. A graceful 7-coloring of TLX_7

Case 3: $n \geq 8$

Since $TLX_8 \subseteq TLX_n$ for every $n \geq 8$, it suffices to examine TLX_8. Suppose that TLX_8 admits a graceful 7-coloring f. Consider the restriction of f to TLX_7. A direct examination of all feasible boundary assignments shows that $f(u_7) = 4$ and the three edges incident with u_7 in TLX_7 receive the induced colors $1, 2, 3$. In TLX_8, the two additional edges u_7w_7 and u_7u_8 become incident with u_7. These edges must receive two new distinct induced colors. However, for every available color $c \in \{1, 2, \ldots, 7\} \backslash \{4\}$, we have $|f(u_7) - c| = |4 - c| \in \{1, 2, 3\}$. All these induced colors have already been used on the edges incident with u_7. Hence neither u_7w_7 nor u_7u_8 can receive a new induced color. This contradiction shows that TLX_8 admits no graceful 7-coloring. Therefore,

$$\chi_g(TLX_n) \geq \chi_g(TLX_8) \geq 8.$$

For the upper bound, define $f: \mathrm{V}(TLX_n) \to \{1, 2, \ldots, 8\}$ by

$i\ (mod\ 3)$	1	2	0
$f(u_i)$	1	2	6
$f(v_i)$	7	4	3
$f(w_i)$	5	7	8

where $1 \leq i \leq n$ for u_i, v_i and $1 \leq i \leq n - 1$ for w_i. Direct verification shows that adjacent vertices receive distinct colors and that the induced colors of the edges incident with each vertex are pairwise

distinct. Thus, f is a graceful 8-coloring of TLX_n, implying that $\chi_g(TLX_n) \leq 8$. Combining both bounds gives $\chi_g(TLX_n) = 8 \quad \text{for } n \geq 8$.

Theorem 3.2. Let $TLXJ_n$ be the bridge triangular ladder graph, where $n \geq 2$. Then

$$\chi_g(TLXJ_n) = \begin{cases} 6, & n = 2 \\ 7, & 3 \leq n \leq 5 \\ 8, & n \geq 6 \end{cases}$$

Proof. We determine the graceful chromatic number of $TLXJ_n$ by establishing the corresponding lower and upper bounds. The proof is divided into three cases: $n = 2$, $3 \leq n \leq 5$, and $n \geq 6$.

Case 1: $n = 2$.

The graph $TLXJ_2$ contains three pairwise adjacent vertices $u_1, u_2,$ and v_2, each of degree four. Suppose that $TLXJ_2$ admits a graceful 5-coloring. Since four distinct induced colors are required on the edges incident with every vertex of degree four, such a vertex must receive color 1 or 5. Therefore,

$$f(u_1), f(u_2), f(v_2) \in \{1, 5\}.$$

However,

$$u_1u_2, \qquad u_1v_2, \qquad u_2v_2 \in E(TLXJ_2),$$

so these three vertices form a triangle. These three pairwise adjacent vertices cannot be properly colored using only the two colors 1 and 5. Thus, $TLXJ_2$ does not admit a graceful 5-coloring, and hence

$$\chi_g(TLXJ_2) \geq 6.$$

For the upper bound, define $f: V(TLXJ_2) \to \{1, 2, \ldots, 6\}$ by

Vertex	u_1	u_2	v_1	v_2	v_3	w_1	w_2
f	6	1	3	2	3	5	4

The corresponding induced edge colors are

$$\begin{array}{llll} f^*(u_1u_2) = 5, & f^*(u_1w_1) = 1, & f^*(u_1v_1) = 3, & f^*(v_1w_1) = 2, \\ f^*(u_1v_2) = 4, & f^*(w_1v_2) = 3, & f^*(u_2w_2) = 3, & f^*(u_2v_2) = 1, \\ f^*(v_2w_2) = 2, & f^*(u_2v_3) = 2, & f^*(w_2v_3) = 1. & \end{array}$$

Adjacent vertices receive distinct colors, and all edges incident with a common vertex receive pairwise distinct induced colors. Hence, f is a graceful 6-coloring of $TLXJ_2$, and $\chi_g(TLXJ_2) \leq 6$. Consequently, $\chi_g(TLXJ_2) = 6$.

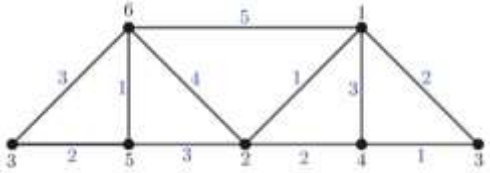


Figure 6. A graceful 6-coloring of $TLXJ_2$

Case 2: $3 \leq n \leq 5$

Since $TLXJ_3 \subseteq TLXJ_n$ every $3 \leq n \leq 5$, it suffices to establish the lower bound for $TLXJ_3$. Suppose that $TLXJ_3$ admits a graceful 6-coloring f. Since $\deg(u_2) = 5$, the five edges incident with u_2 must receive the five distinct induced colors $1, 2, 3, 4, 5$. Hence, $f(u_2) \in \{1, 6\}$. Replacing each color c with $7 - c$, if necessary, preserves every induced edge color. Thus, we may assume that $f(u_2) = 1$. Therefore, its five neighbors must receive the colors $2, 3, 4, 5,$ and 6, so

$$\{f(u_1), f(v_2), f(w_2), f(v_3), f(u_3)\} = \{2, 3, 4, 5, 6\}.$$

None can be extended to w_1, v_1, w_3, and v_4. Hence, $TLXJ_3$ admits no graceful 6-coloring. Since $TLXJ_3 \subseteq TLXJ_4 \subseteq TLXJ_n$, it follows that $\chi_g(TLXJ_n) \geq 7$ for $3 \leq n \leq 5$

For the upper bound, : $f: V(TLXJ_n) \to \{1, 2, \ldots, 7\}$ by the following assignments:

n	$(f(u_1),. \ . \ ., f(u_n))$	$(f(w_1),. \ . \ ., f(w_n))$	$(f(v_1),. \ . \ ., f(v_{n+1}))$
3	$(2, 1, 6)$	$(4, 7, 2)$	$(7, 5, 3, 5)$
4	$(6, 1, 2, 6)$	$(2, 7, 4, 1)$	$(5, 3, 5, 7, 3)$
5	$(6, 2, 1, 6, 2)$	$(1, 4, 7, 4, 1)$	$(3, 7, 5, 3, 7, 4)$

For each assignment, adjacent vertices receive distinct colors, and the induced color of all edges incident with a common vertex are pair. Direct verification shows that each assignment is a graceful 7-coloring. Thus, $\chi_g(TLXJ_n) \leq 7,\ 3 \leq n \leq 5$. Combining the lower and upper bounds yields $\chi_g(TLXJ_n) = 7,\ 3 \leq n \leq 5$.

Case 3: $n \geq 6$.
Since $TLXJ_6 \subseteq TLXJ_n$ for every $n \geq 6$, it suffices to prove that $\chi_g(TLXJ_6) \geq 8$. Suppose that $TLXJ_6$ admits a graceful 7-coloring f. Since $d(u_i) = 5$ for $2 \leq i \leq 5$, the five edges incident with each u_i must receive distinct induced colors. Therefore,

$$f(u_i) \in \{1, 2, 6, 7\}, \qquad 2 \leq i \leq 5,$$

because a vertex assigned one of the colors $3, 4,$ and 5 cannot generate five distinct positive differences from $\{1,. \ . \ ., 7\}$. A direct examination of all assignments satisfying these restrictions and the proper vertex-coloring and induced edge-coloring conditions shows that none can be extended to all vertices of $TLXJ_6$. Hence, $TLXJ_6$ admits no graceful 7-coloring. Therefore,

$$\chi_g(TLXJ_n) \geq \chi_g(TLXJ_6) \geq 8.$$

For the upper bound, define $f: V(TLXJ_n) \to \{1, 2, \ldots, 8\}$ by

$$f(u_i) = \begin{cases} 1, & i \equiv 1\ (mod\ 3), \\ 2, & i \equiv 2\ (mod\ 3), \\ 6, & i \equiv 0\ (mod\ 3), \end{cases}$$

$$f(w_i) = \begin{cases} 8, & i \equiv 1\ (mod\ 3), \\ 5, & i \equiv 2\ (mod\ 3), \\ 4, & i \equiv 0\ (mod\ 3), \end{cases}$$

and

$$f(v_i) = \begin{cases} 3, & i \equiv 1 \ (mod\ 3), \\ 4, & i \equiv 2 \ (mod\ 3), \\ 7, & i \equiv 0 \ (mod\ 3). \end{cases}$$

Here, $1 \leq i \leq n$ for u_i, w_i and $1 \leq i \leq n+1$ for v_i. The assignment gives different colors to adjacent vertices. Checking the three residue classes modulo 3 shows that all edges incident with a common vertex receive pairwise distinct induced colors. Thus, f is a graceful 8-coloring and hence $\chi_g(TLXJ_n) \leq 8$. We conclude, $\chi_g(TLXJ_n) = 8,\ n \geq 6$.

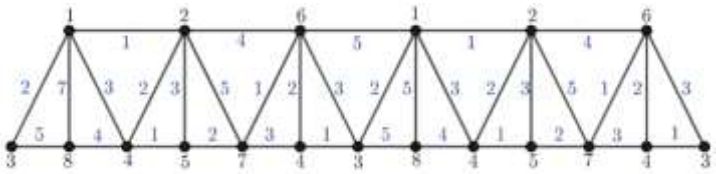


Figure 7. A graceful 8-coloring of $TLXJ_6$

Lemma 3.1

The graph CTL_n admits a graceful 7-coloring if and only if $3 \mid n$. Moreover, every graceful 7-coloring of CTL_n has period 3; that is,

$$f(u_{i+3}) = f(u_i) \quad \text{and} \quad f(v_{i+3}) = f(v_i)$$

for every admissible index i.

Proof. Suppose that f is a graceful 7-coloring of CTL_n. Since every vertex of CTL_n has degree 4, the four edges incident with each vertex must receive four distinct induced colors. Consequently, color 4 cannot be assigned to any vertex because it produces only the three positive differences 1,2, and 3. Hence, $f(x) \in \{1, 2, 3, 5, 6, 7\}$ for every $x \in V(CTL_n)$. The indices are taken modulo n. At u_i, the induced colors

$$|f(u_i) - f(u_{i-1)}|, \quad |f(u_i) - f(u_{i+1})|, \quad |f(u_i) - f(v_i)|, \quad |f(u_i) - f(v_{i+1})|$$

must be pairwise distinct. Similarly at v_i, the induced colors

$$|f(v_i) - f(v_{i-1)}|, \quad |f(v_i) - f(v_{i+1})|, \quad |f(v_i) - f(u_i)|, \quad |f(v_i) - f(u_{i-1})|$$

must be pairwise distinct. Now consider consecutive columns:

$$(u_{i-1}, v_{i-1}), \qquad (u_i, v_i), \qquad (u_{i+1}, v_{i+1}).$$

A complete examination of the finite color assignments from $\{1, 2, 3, 5, 6, 7\}$ satisfying the two pairwise distinctness conditions shows that every closed sequence repeats after three columns. Consequently, $\big(f(u_{i+3}), f(v_{i+3})\big) = \big(f(u_i), f(v_i)\big)$. Hence, every graceful 7-coloring is periodic with period 3. One representative pattern is

$i\ (mod\ 3)$	1	2	0
$f(u_i)$	1	3	7
$f(v_i)$	5	6	2

If $n \equiv 0(\text{mod } 3)$, the periodic assignment is compatible with all closing edges and therefore defines a graceful 7-coloring of CTL_n. If $n \equiv 1(\text{mod } 3)$, at least one closing edge has endpoints with the same color. If $n \equiv 2(\text{mod } 3)$, two closing edges incident with a common vertex receive the same induced color. Therefore, the periodic assignment closes properly if and only if $3 \mid n$. Thus,

$$CTL_n \text{ admits a graceful 7-coloring} \Leftrightarrow 3 \mid n$$

Theorem 3.3

For every integer $n \geq 3$, the graceful chromatic number of the circular triangular ladder graph CTL_n is

$$\chi_g(CTL_n) = \begin{cases} 7, & n \equiv 0 \pmod 3, \\ 8, & n \not\equiv 0 \pmod 3. \end{cases}$$

Proof. We first establish the general lower bound. Suppose that CTL_n admits a graceful 6-coloring f. Since every vertex has degree 4, the four incident edges at each vertex must receive four distinct induced colors. Colors 3 and 4 cannot be assigned to any vertex because each produces at most three distinct positive differences from $\{1, \ldots, 6\}$. Therefore,

$f(x) \in \{1,2,5,6\}$ for every $x \in V(CTL_n)$. However, every color in this set produces at most three distinct differences from the other colors in the same set. Hence, at least two of the four edges incident with a vertex must receive the same induced color, a contradiction. Thus, $\chi_g(CTL_n) \geq 7$.

We now consider two cases.

Case 1: $n \equiv 0(\text{mod } 3)$. Define $f: V(CTL_n) \to \{1, \ldots, 7\}$ by

$$f(u_i) = \begin{cases} 1, & i \equiv 1 \pmod 3, \\ 3, & i \equiv 2 \pmod 3, \\ 7, & i \equiv 0 \pmod 3, \end{cases}$$

and

$$f(v_i) = \begin{cases} 5, & i \equiv 1 \pmod 3, \\ 6, & i \equiv 2 \pmod 3, \\ 2, & i \equiv 0 \pmod 3. \end{cases}$$

The induced colors incident with u_i and v_i are

$i \pmod 3$	Induced colors at u_i	Induced colors at v_i
1	$\{2, 4, 5, 6\}$	$\{1, 2, 3, 4\}$
2	$\{1, 2, 3, 4\}$	$\{1, 3, 4, 5\}$
0	$\{2, 4, 5, 6\}$	$\{1, 3, 4, 5\}$

Each set contains four distinct values. Since $3 \mid n$, the assignment is also compatible with the closing edges. Therefore, f is a graceful 7-coloring, and $\chi_g(CTL_n) \leq 7$. Together with the lower bound, this gives

$$\chi_g(CTL_n) = 7, \qquad n \equiv 0 \pmod 3.$$

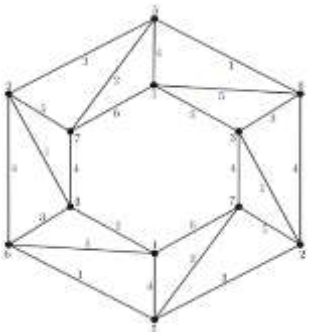


Figure 8. A graceful 7-coloring of CTL_6

Case 2: $n \not\equiv 0 \pmod 3$

By Lemma 3.1, a graceful 7-coloring cannot be closed when $n \not\equiv 0 \pmod 3$. Therefore, $\chi_g(CTL_n) \geq 8$. It remains to construct graceful 8-colorings.

If $n \equiv 1 \pmod 3$, use the periodic assignment from Case 1 for $1 \leq i \leq n-2$, and define:

	$n-1$	n
$f(u_i)$	4	2
$f(v_i)$	8	7

If $n \equiv 2 \pmod 3$, use the periodic assignment from Case 1 for $1 \leq i \leq n-3$, and define:

	$n-2$	$n-1$	n
$f(u_i)$	8	6	2
$f(v_i)$	4	1	7

In both subcases, adjacent vertices receive different colors. Moreover, the induced colors of the edges incident with each modified vertex and each endpoint of a closing edge are pairwise distinct. The unchanged vertices retain the graceful-coloring conditions of the period-three assignment. Hence, both constructions are graceful 8-colorings, and $\chi_g(CTL_n) \leq 8$. Consequently, $\chi_g(CTL_n) = 8, \quad n \not\equiv 0 \pmod 3$.

Therefore,

$$\chi_g(CTL_n) = \begin{cases} 7, & n \equiv 0 \pmod 3, \\ 8, & n \not\equiv 0 \pmod 3. \end{cases}$$

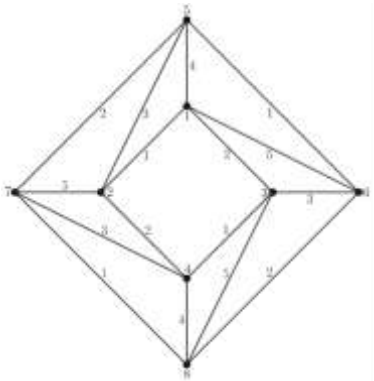


Figure 9. A graceful 8-coloring of CTL_4

D. Conclusion

This study determines the graceful chromatic numbers of three classes of triangular ladder graphs. The results are

$$\chi_g(TLX_n) = \begin{cases} 5, & n = 2 \\ 7, & 3 \le n \le 7 \\ 8, & n \ge 8 \end{cases} \qquad \chi_g(TLXJ_n) = \begin{cases} 6, & n = 2 \\ 7, & 3 \le n \le 5 \\ 8, & n \ge 6 \end{cases}$$

Furthermore, for every integer $n \ge 3$,

$$\chi_g(CTL_n) = \begin{cases} 7, & n \equiv 0 \pmod 3, \\ 8, & n \not\equiv 0 \pmod 3. \end{cases}$$

These results show that the graceful chromatic numbers are determined by both the order and structure of the graphs. For TLX_n, the value becomes constant at 8 for $n \ge 8$, while that of $TLXJ_n$ becomes constant at 8 for $n \ge 6$. In contrast, the graceful chromatic number of CTL_n depends on the residue of n modulo 3. Future studies may investigate graceful chromatic numbers for other ladder-graph variants and graphs obtained through different graph operations.

Pernyataan Penggunaan Kecerdasan Artifisial

Penulis menggunakan ChatGPT (OpenAI) untuk membantu menyunting bahasa dan tata tulis naskah. Seluruh keluaran telah diperiksa dan diverifikasi oleh penulis, dan penulis bertanggung jawab penuh atas isi artikel.

Pernyataan Kontribusi Penulis

Raphita Yanisari Silalahi	*Konseptualisasi penelitian, pengembangan konstruksi pewarnaan dan pembuktian matematis, serta penyusunan draf awal naskah.*
I Nengah Suparta	*Pengarahan metodologi, verifikasi pembuktian matematis dan penyempurnaan naskah.*
Sariyasa	*Validasi hasil dan penyempurnaan naskah.*
Kadek Ria Andriani	*Pemeriksaan konstruksi pewarnaan.*
Rasmin Siringo-ringo	*Pemeriksaan contoh pewarnaan, perapian notasi dan ilustrasi graf.*

Pernyataan Pendanaan

Penelitian ini tidak menerima pendanaan dari lembaga mana pun, baik pemerintah, swasta, maupun nirlaba.

Pernyataan Konflik Kepentingan

Penulis menyatakan tidak terdapat konflik kepentingan terkait penelitian, kepenulisan, maupun penerbitan artikel ini.

Daftar Pustaka

Alfarisi R., Dafik, Prihandini R. M., Adawiyah R., Albirri E. R., Agustin I. H. (2019). Graceful Chromatic Number of Unicyclic Graphs. Journal of Physics: Conference Series, 1306(1), 012039. https://doi.org/10.1088/1742-6596/1306/1/012039

Asyari M. L., Agustin I. H., Nisviasari R., Adawiyah R. (2022). On graceful chromatic number of some graphs. Journal of Physics: Conference Series, 2157(1), 012013. https://doi.org/10.1088/1742-6596/2157/1/012013

Atmadja, K., & Marhaeni, M. (2020). Pelabelan Harmonis Pada Graf Tangga Segitiga Jembatan Xj_n. PRISMA, Prosiding Seminar Nasional Matematika, 3, 25-28. https://journal.unnes.ac.id/sju/prisma/article/view/37613

Atmadja, K., & Sugeng, K.A. (2017). Pelabelan Harmonis Pada Graf Tangga Segitiga Variasi, In Prosiding Seminar Nasional Matematika 2017 (pp. 642-647). Departemen Matematika, FMIPA, Universitas Indonesia.

Bi, Z., Byers, A., English, S., Laforge, E., Zhang, P., (2017), Graceful colorings of graphs, Journal of Combinatorics. Mathematics and Combinatorics Computing, 101, 101-119.

Bondy, J. A., & Murty, U. S. R. (1976). Graph Theory with Applications. Macmillan.

Chartrand, G. dan Zhang, P. (2009). Chromatic Graph Theory. CRC Press.

Gallian, J. A. (2018). A dynamic survey of graph labeling. The Electronic Journal of Combinatorics, 1(Dynamic Surveys), Article DS6.

Golomb, S. W. (1972). How to number a graph. In R. C. Read (Ed.), Graph Theory and Computing (pp. 23–37). New York: Academic Press. https://doi.org/10.1016/B978-1-4832-3187-7.50008-8

Kristiana A. I, Anzori, A., Adawiyah, R., Slamin, & Albirri, E. R. (2023). Bilangan Kromatik Graceful Pada Keluarga Graf Grid. Jurnal Axioma : Jurnal Matematika dan Pembelajaran, 8(2), 144-155. https://doi.org/10.56013/axi.v8i2.1544

Laavanya, D., & Yamini, S. D. (2024). Graceful coloring of ladder graphs. TWMS Journal of Applied and Engineering Mathematics, 14(3), 991–1001.

Lestari, D. D. A., Kristiana, Prihandini R. M., Alfarisi R., Setiawan T.B., Adawiyah R. (2024). Bilangan Kromatik Graceful pada Keluarga Graf Sentripetal. Journal Diferensial, 6(1), 57-64. https://doi.org/10.35508/jd.v6i1.12746

Lestari, D. (2022). Nilai Total Ketidakteraturan Titik Graf Tangga Segitiga Melingkar yang dimodifikasi dengan 4n Titik untuk n Ganjil [Undergraduate thesis, Universitas Hasanuddin].

Mincu R., Obreja C., Popa A. (2019). The graceful chromatic number for some particular classes of graphs. In 2019 21st International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC). IEEE. https://doi.org/10.1109/SYNASC49474.2019.00024

Rosa, A. (1967). On certain valuations of the vertices of a graph. In Theory of Graphs: International Symposium, Rome, July 1966 (pp. 349–355). New York: Gordon and Breach.

Suparta, I Nengah., Venkathacalam, M., Aris Gunadi, I. G. A., & Pratama, P. A. C. (2023). Graceful Chromatic Number of Some Cartesian Product Graphs. Ural Mathematical Journal, 9(2), 193-208. https://doi.org/10.15826/umj.2023.2.016